Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Prove that in any acute triangle $\triangle ABC$ with the sides a,b,c holds inequality

$$27 \leq (a+b+c)^2\left(\frac{1}{a^2+b^2-c^2}+\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}\right).$$

Solution.

Let R,r,s be circumradius, inradius and semiperimeter of $\triangle ABC$, respectively, then using cosine-theorem, sine-theorem and correlation $abc = 4Rrs$ we obtain

$$27 \leq (a+b+c)^2\left(\frac{1}{a^2+b^2-c^2}+\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}\right) \Leftrightarrow$$

$$\frac{27}{4s^2} \leq \frac{1}{2ab\cos C}+\frac{1}{2bc\cos A}+\frac{1}{2ca\cos B} \Leftrightarrow \frac{27abc}{2s^2} \leq \frac{a\tan A}{\sin A}+\frac{b\tan B}{\sin B}+\frac{c\tan C}{\sin C} \Leftrightarrow$$

$$\frac{27 \cdot 4Rrs}{2s^2} \leq 2R(\tan A+\tan B+\tan C) \Leftrightarrow \frac{27r}{s} \leq \tan A+\tan B+\tan C = \tan A\tan B\tan C.$$

Since, by AM-GM inequality

$3\sqrt[3]{\tan A\tan B\tan C} \leq \tan A+\tan B+\tan C = \tan A\tan B\tan C$ then $3\sqrt{3} \leq \tan A\tan B\tan C$.

Thus, it is enough to prove that $\frac{27r}{s} \leq 3\sqrt{3} \Leftrightarrow 3\sqrt{3}\,r \leq s \Leftrightarrow$

$$6\sqrt{3}\,r \leq a+b+c.$$

Applying again AM-GM inequality and Euler's inequality $R \geq 2r$ we obtain

$$2s = a+b+c \geq 3\sqrt[3]{abc} = 3\sqrt[3]{4Rrs} \Leftrightarrow 8s^3 \geq 27 \cdot 4Rrs \Leftrightarrow$$

$$2s^2 \geq 27Rr \Rightarrow 2s^2 \geq 27 \cdot 2r^2 \Leftrightarrow s \geq 3\sqrt{3}\,r \Leftrightarrow 6\sqrt{3}\,r \leq a+b+c.$$